IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

The asymptotic behaviour of matrix superpropagators in quantum gravity

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1978 J. Phys. A: Math. Gen. 11 885

(http://iopscience.iop.org/0305-4470/11/5/019)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 30/05/2010 at 18:51

Please note that terms and conditions apply.

J. Phys. A: Math. Gen., Vol. 11, No. 5, 1978. Printed in Great Britain

The asymptotic behaviour of matrix superpropagators in quantum gravity

H E De Meyer†

Seminarie voor Wiskundige Natuurkunde, Rijksuniversiteit-Gent, Krijgslaan 271-S9, B-9000 Gent, Belgium

Received 10 November 1977, in final form 21 December 1977

Abstract. In the context of quantum gravity, the asymptotic behaviour of exponential multi-matrix superpropagators is investigated. By varying the dimension of the matrix field, a remarkable property of two-point functions is found, which suggests that quantum gravity is ambiguity free.

1. Introduction

In non-linear chiral theories and gravity modified field theories, which involve Lagrangians depending on matrix fields, one needs as a basic tool for calculations the vacuum expectation values of time-ordered products of scalar and matrix functions of these fields. Delbourgo (1972) used Fourier transform methods to evaluate superpropagators for fields with isospin. By means of superpropagator calculations Isham *et al* (1971) proved the regulating effect of rational parametrised gravity on quantum electrodynamics, and they afterwards demonstrated the advantages of a localisable (i.e. exponential) parametrisation of the gravitational interaction (Isham *et al* 1972). In the latter case one needs the superpropagator

$$\mathscr{E}^{(4)}_{1,1}[\Delta] \equiv \langle {e^{\kappa\phi(x)}}_{\alpha\beta}, {e^{\kappa\phi(0)}}_{\gamma\delta}\rangle, \tag{1.1}$$

where the symmetrical graviton field $\phi_{\alpha\beta}(x)$ propagates as

$$\langle\phi_{\alpha\beta}(x)\phi_{\gamma\delta}(0)\rangle = \tfrac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - 2c\eta_{\alpha\beta}\eta_{\gamma\delta})\Delta(x). \tag{1.2}$$

Herein $\Delta(x)$ denotes the zero-mass scalar propagator, η is the Minkowski metric, κ is an arbitrary constant and c is the so called gauge parameter. It has been pointed out by Isham *et al* (1973) that the form (1.2) for the bare graviton propagator is valid only when $c = \omega(\omega-1)+\frac{1}{2}$, where ω is the weight of the contravariant tensor density $\hat{g}^{\mu\nu}$ parametrised in the form

$$\hat{g}^{\mu\nu} \equiv (\sqrt{-g})^{\omega} g^{\mu\nu} = (e^{\kappa\phi})^{\mu\nu}.$$

An analytical expression for $\mathscr{E}^{(4)}_{1,1}[\Delta]$ was first obtained by Ashmore and Delbourgo (1973a) by use of an integral representation of an arbitrary power of the determinant of a matrix, originally due to von Siegel (1935). They also found that $\mathscr{E}^{(4)}_{1,1}[\Delta]$ is an entire function of Δ. They showed that the leading behaviour of that function, defined

† Aangesteld navorser bij het Nationaal Fonds voor Wetenschappelijk Onderzoek (Belgium).

0305-4770/78/0005-0885$02.00 © 1978 The Institute of Physics

by (1.1), as $\Delta \to +\infty$ is given by

$$\mathscr{E}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{3/2}\, e^{\kappa^2(1-c)\Delta(x)}, \tag{1.3}$$

whereby α, β, γ and δ can take any value running from 1 to 4. It has further been demonstrated that the Ashmore technique can be enlarged to include the case of chiral SU(3) superpropagators (Ashmore and Delbourgo 1973b). An alternative method for the evaluation of such superpropagators has been given by Kapoor (1976).

Defining two-point amplitudes $\mathscr{E}^{(4)}_{k,l}[\Delta]$ by

$$\mathscr{E}^{(4)}_{k,l}[\Delta] \equiv \left\langle \prod_{i=1}^{k} e^{\kappa\phi(x)}{}_{\alpha_i\beta_i}, \prod_{j=1}^{l} e^{\kappa\phi(0)}{}_{\gamma_j\delta_j} \right\rangle, \tag{1.4}$$

it has been shown (De Meyer 1976) that the Ashmore technique can be used to calculate all propagators of the form $\mathscr{E}^{(4)}_{1,l}[\Delta]$ or $\mathscr{E}^{(4)}_{k,1}[\Delta]$. By explicit calculation of $\mathscr{E}^{(4)}_{1,2}[\Delta]$ it was found that

$$\mathscr{E}^{(4)}_{1,2}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{3/2}\, e^{2\kappa^2(1-c)\Delta(x)}, \tag{1.5}$$

as $\Delta(x)$ tends to infinity, whereas it was also argued that in the same limit, $\mathscr{E}^{(4)}_{1,l}[\Delta]$ behaves as:

$$\mathscr{E}^{(4)}_{1,l}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{3/2}\, e^{l\kappa^2(1-c)\Delta(x)}. \tag{1.6}$$

In the present paper the leading asymptotic behaviour of the two-point amplitudes $\mathscr{E}^{(4)}_{k,l}[\Delta]$ is investigated. In particular an argument is given which shows that the traced superpropagators $\mathscr{T}^{(4)}_{k,l}[\Delta]$ defined by

$$\mathscr{T}^{(4)}_{k,l}[\Delta] \equiv \left\langle \prod_{i=1}^{k} \operatorname{Tr} e^{\kappa\phi(x)}, \prod_{j=1}^{l} \operatorname{Tr} e^{\kappa\phi(0)} \right\rangle, \tag{1.7}$$

behave well for $c > 1$, in the sense that for such c values $\mathscr{T}^{(4)}_{k,l}[\Delta]$ is a finite distribution, free of ambiguities. Furthermore, we analyse the asymptotic behaviour of the functions $\mathscr{T}^{(\nu)}_{k,l}[\Delta]$ for some ν values, where ν refers to the dimensions of the matrix field $\phi_{\alpha\beta}(x)$. From this we get an indication that the question of Isham *et al* (1973), as to whether quantum gravity is ambiguity free, can be affirmatively answered.

2. An asymptotic limiting procedure

By the use of the Ashmore technique (Ashmore and Delbourgo 1973a) one easily obtains the following integral representation:

$$\langle \operatorname{Tr} \phi^N(x), \operatorname{Tr} \phi^N(0)\rangle = N\Delta^N(x) \frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2} \int_{\mathscr{D}} \frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)} e^{-\operatorname{Tr} X} \operatorname{Tr}[(X - c\operatorname{Tr} X)^N]. \tag{2.1}$$

Consequently, the trace superpropagator $\mathscr{T}^{(4)}_{1,1}[\Delta]$ can be written as

$$\mathscr{T}^{(4)}_{1,1}[\Delta] = \frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2} \int_{\mathscr{D}} \frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)} e^{-\operatorname{Tr} X} \operatorname{Tr}\left(\sum_{N=1}^{\infty} \frac{[\kappa^2\Delta(x)(X - c\operatorname{Tr} X)]^N}{N!(N-1)!} \right). \tag{2.2}$$

The series in the right-hand side of (2.2) can be summed, and one finds

$$\mathscr{T}^{(4)}_{1,1}[\Delta] = \frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2} \int_{\mathscr{D}} \frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)} e^{-\mathrm{Tr}\,X} \times \mathrm{Tr}[\![\sqrt{\kappa^2\Delta(x)(X - c\,\mathrm{Tr}\,X)}\{I_1[2\sqrt{\kappa^2\Delta(x)(X - c\,\mathrm{Tr}\,X)}]\}]\!], \tag{2.3}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. The leading term of the asymptotic expansion of $I_\nu(z)$ is given by (Gradshteyn and Ryzhik 1965):

$$I_\nu(z) \underset{|z|\to\infty}{\sim} e^z/\sqrt{z}. \tag{2.4}$$

Therefore, one obtains from (2.3)

$$\mathscr{T}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} \frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2} \int_{\mathscr{D}} \frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)} e^{-\mathrm{Tr}\,X} \times \mathrm{Tr}\,\{[\kappa^2\Delta(x)(X - c\,\mathrm{Tr}\,X)]^{1/4}\, e^{2\sqrt{\kappa^2\Delta(x)(X - c\,\mathrm{Tr}\,X)}}\}. \tag{2.5}$$

Since the matrix X can be diagonalised in the integrand of the representation (2.2), one can, without loss of generality, regard X in (2.5) as being diagonal. One also introduces the new variables $\Lambda = \mathrm{Tr}\,X$, and three dimensionless arbitrary variables depending on the elements of the diagonal matrix X. The matrix X occurs twice in the integrand of (2.5) in the combination $X\Delta(x)$. It is clear that the principal contribution to $\mathscr{T}^{(4)}_{1,1}[\Delta]$ as $\Delta \to +\infty$ comes from these X in the product $X\Delta(x)$, whereby one of the positive diagonal elements reaches the maximum value Λ, the other elements then being zero. As a consequence, $X\Delta(x)$ in (2.5) can be replaced by $\Lambda\Delta(x)$. Finally one suggests the following substitution rule:

$$\frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2} \int_{\mathscr{D}} \frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)} \to \int_0^\infty \rho(\Lambda)\,d\Lambda, \tag{2.6}$$

to obtain a formula for the leading term of the asymptotic expansion of $\mathscr{T}^{(4)}_{1,1}[\Delta]$. In (2.6) $\rho(\Lambda)$ is a so far undetermined weight function. From the foregoing it follows that

$$\mathscr{T}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} \int_0^\infty d\Lambda\,\rho(\Lambda)[\kappa^2(1-c)\Lambda\Delta(x)]^{1/4}\, e^{2\sqrt{\kappa^2(1-c)\Lambda\Delta(x)}}. \tag{2.7}$$

By putting $\Lambda = t^2$, and making the choice $\rho(t^2) = t^\gamma$, with γ a constant, for the weight function, (2.7) transforms to

$$\mathscr{T}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} [\kappa^2(1-c)\Delta(x)]^{1/4} \int_0^\infty dt\, t^{\frac{3}{2}+\gamma}\, e^{-t^2}\, e^{2t\sqrt{\kappa^2(1-c)\Delta(x)}},$$

or after carrying out the integration:

$$\mathscr{T}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} [\kappa^2(1-c)\Delta(x)]^{1/4}\, e^{\frac{1}{2}\kappa^2(1-c)\Delta(x)}\, D_{-\frac{5}{2}-\gamma}[-\sqrt{2\kappa^2(1-c)\Delta(x)}]. \tag{2.8}$$

Herein $D_p(z)$ is a parabolic cylinder function, of which the asymptotic expansion is given by (Gradshteyn and Ryzhik 1965):

$$D_p(z) \underset{|z|\to\infty}{\sim} e^{-z^2/4} z^p\Big(1 + O\Big(\frac{1}{z^2}\Big)\Big) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i}\, e^{z^2/4} z^{-p-1}\Big(1 + O\Big(\frac{1}{z^2}\Big)\Big), \qquad \Big(\frac{\pi}{4} < \arg z < \frac{5\pi}{4}\Big).$$

Substitution of the leading term in (2.8) leads to

$$\mathscr{T}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{1+\frac{1}{2}\gamma}\, e^{\kappa^2(1-c)\Delta(x)}, \tag{2.9}$$

which is in perfect agreement with the result (1.3) if one chooses $\gamma = 1$. The substitution rule (2.6) is now brought into the form:

$$\frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2} \int_{\mathscr{D}} \frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)} \to \int_0^{\infty} \Lambda^{1/2}\, d\Lambda. \tag{2.10}$$

By direct calculation one can verify that naive changing of X to $\operatorname{Tr} X$ in the representation (2.2) does not reproduce the correct asymptotic behaviour of $\mathscr{T}^{(4)}_{1,1}[\Delta]$. The fact that the weight function in (2.10) turns out to be different from unity demonstrates that the integrations with respect to the three dimensionless variables contribute to the asymptotic behaviour. These integrations are similar in all cases whereby the superpropagator can be expressed as a single integral of the Siegel-type. Therefore (2.10) can be used to recover the asymptotic properties of the functions $\mathscr{T}^{(4)}_{1,l}[\Delta]$ or $\mathscr{T}^{(4)}_{k,1}[\Delta]$.

In order to calculate the leading term in the asymptotic expansion of $\mathscr{T}^{(4)}_{1,l}[\Delta]$, one starts from the following expression:

$$\left\langle \operatorname{Tr}\phi^L(x), \prod_{j=1}^{l} \operatorname{Tr}\phi^{M_j}(0)\right\rangle = L\Delta^L(x)\frac{\partial}{\partial\mu}\bigg|_{\mu=-5/2}\int_{\mathscr{D}}\frac{dX|X|^{\mu}}{\pi^3\Gamma_4(\mu)}\, e^{-\operatorname{Tr} X}\prod_{j=1}^{l}\operatorname{Tr}[(X-c\operatorname{Tr} X)^{M_j}],$$
$$\left(\sum_{j=1}^{l} M_j = L\right). \tag{2.11}$$

By use of the limiting procedure, one obtains

$$\left\langle \operatorname{Tr}\phi^L(x), \prod_{j=1}^{l} \operatorname{Tr}\phi^{M_j}(0)\right\rangle \underset{\Delta\to+\infty}{\sim} L\Delta^L(x)\int_0^{\infty} d\Lambda\, \Lambda^{1/2}\, e^{-\Lambda}[(1-c)\Lambda]^L$$
$$= L[(1-c)\Delta(x)]^L\Gamma(L+\tfrac{3}{2}), \qquad \left(\sum_{j=1}^{l} M_j = L\right),$$

from which one finds after a short calculation the following behaviour of $\mathscr{T}^{(4)}_{1,l}[\Delta]$ as $\Delta \to +\infty$:

$$\mathscr{T}^{(4)}_{1,l}[\Delta] \underset{\Delta\to+\infty}{\sim} \sum_{L=0}^{\infty}\sum_{M_1=0}^{\infty}\cdots\sum_{M_l=0}^{\infty} \frac{L[\kappa^2(1-c)\Delta(x)]^L}{L!M_1!\dots M_l!}\Gamma(L+\tfrac{3}{2})\delta_{L,\Sigma_{j=1}^{l}M_j}$$
$$= \sum_{L=0}^{\infty} \frac{L[\kappa^2(1-c)\Delta(x)]^L}{L!^2}\Gamma(L+\tfrac{3}{2})l^L.$$

It is easily proved that

$$\sum_{L=1}^{\infty} \frac{\lambda^L\Gamma(L+\frac{3}{2})}{\Gamma(L+1)\Gamma(L)} = \frac{3\lambda\sqrt{\pi}}{4}\,{}_1F_1(\tfrac{5}{2}; 2; \lambda), \tag{2.12}$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometrical function. The asymptotic expansion of this function is given by

$$ {}_1F_1(a;b;z) \underset{|z|\to\infty}{\sim} \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z g(a-1;b-1;z) + \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a} g(a;a-b+1;z), $$

with

$$ g(\alpha;\beta;z) = 1 + \frac{\alpha\beta}{1!z} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!z^2} + \dots, $$

and therefore, one finally obtains

$$ \mathcal{T}^{(4)}_{1,l}[\Delta] \underset{\Delta\to+\infty}{\sim} [\kappa^2\Delta(x)]^{3/2} e^{l\kappa^2(1-c)\Delta(x)}. \tag{2.13} $$

This behaviour is in perfect agreement with the expected asymptotic behaviour (1.6) of $\mathscr{E}^{(4)}_{1,l}[\Delta]$. It is also obvious that $\mathscr{E}^{(4)}_{k,1}[\Delta]$ behaves in the same way as $\mathscr{E}^{(4)}_{1,k}[\Delta]$ as $\Delta \to +\infty$, and one thus has a confirmation that for $c > 1$ the distributions $\mathscr{E}^{(4)}_{k,1}[\Delta]$ and $\mathscr{E}^{(4)}_{1,l}[\Delta]$ are free of ambiguities.

3. Asymptotic behaviour for variable field dimension

We now turn to the problem of evaluating the leading term in the asymptotic expansion of the functions $\mathscr{E}^{(4)}_{k,l}[\Delta]$ with $k > 1$ and $l > 1$. Since in these cases the Ashmore technique leads to multiple integrals of the Siegel-type, it is not possible to apply the limiting procedure of § 2. Instead, we try to make an estimate of the leading term by the following dimensional analysis.

Looking at the trace superpropagator $\mathcal{T}^{(\nu)}_{1,1}[\Delta]$, where ν stands for the dimension of the field $\phi_{\alpha\beta}(x)$, one notices that the calculation of this superpropagator is almost trivial for $\nu = 1$. Indeed, one easily finds

$$ \mathcal{T}^{(1)}_{1,1}[\Delta] \equiv \langle e^{\kappa\phi(x)}, e^{\kappa\phi(0)} \rangle = e^{\kappa^2(1-c)\Delta(x)}, \tag{3.1} $$

with the help of (1.2) which in the present case reduces to

$$ \langle \phi(x), \phi(0) \rangle = (1-c)\Delta(x). $$

Obviously

$$ \mathcal{T}^{(1)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^0 e^{\kappa^2(1-c)\Delta(x)}. \tag{3.2} $$

Comparing this to the leading term in the asymptotic expansion of $\mathcal{T}^{(4)}_{1,1}[\Delta]$:

$$ \mathcal{T}^{(4)}_{1,1}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{3/2} e^{\kappa^2(1-c)\Delta(x)}, \tag{3.3} $$

one remarks that the exponential factor is the same, whereas the accompanying power of $\Delta(x)$ is not.

In the case $\nu = 2$, the superpropagator $\mathscr{E}^{(2)}_{1,1}[\Delta]$ can be evaluated in exactly the same way as for $\nu = 4$. Defining for positive natural numbers ν coefficients $a^{(\nu)}_N$ and functions $a^\nu[\Delta]$ by:

$$ \langle \mathrm{Tr}\,\phi^N(x), \mathrm{Tr}\,\phi^N(0) \rangle = N!\,\nu a^{(\nu)}_N \Delta^N(x), \tag{3.4} $$

$$ a^{(\nu)}[\Delta] = \sum_{N=0}^{\infty} \frac{(\kappa^2\Delta(x))^N}{N!} a^{(\nu)}_N, \tag{3.5} $$

whereby $\phi_{\alpha\beta}(x)$ is a (ν, ν) matrix field satisfying (2.1), one finally obtains in a straightforward manner

$$a^{(2)}[\Delta] = [e^{z(1-2c)}(2+\pi z \mathbf{L}_0(z))]|_{z=\kappa^2\Delta(x)/2}. \tag{3.6}$$

Herein, $\mathbf{L}_0(z)$ is a modified Struve function, which behaves asymptotically as

$$\mathbf{L}_0(z) \underset{z\to\infty}{\sim} e^z/\sqrt{z}. \tag{3.7}$$

By combining (3.4)–(3.7) it is easily demonstrated that

$$\mathcal{T}_{1,1}^{(2)}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{1/2} e^{\kappa^2(1-c)\Delta(x)}. \tag{3.8}$$

In this expression one recovers the same exponential as in (3.2) and (3.3).

Finally $\mathscr{E}_{1,1}^{(3)}[\Delta]$ can be evaluated by slightly modifying the Ashmore technique, which in its original form is only valid for even dimensions of the matrix field (Ashmore and Delbourgo 1973a). It turns out that for odd dimensions the calculations become much more involved, which is an indirect manifestation of the nontrivial and fundamental differences between even and odd fields, already observed by Delbourgo (1972). We only give here the result for $a^{(3)}[\Delta]$, which by adaption of the notations, is the same as that quoted in a footnote of a paper by Ashmore and Delbourgo (1973b):

$$a^{(3)}[\Delta] = \{\tfrac{2}{3}z\, e^{-2cz}(E_i(2z) - 2E_i(z) + \ln\tfrac{1}{2}z + \gamma - 2) + \tfrac{1}{3}e^{z(1-2c)}[(4c-3)\sinh z + (4z+5)\cosh z + 4]\}|_{z=\kappa^2\Delta(x)/2}. \tag{3.9}$$

Herein $E_i(z)$ is the exponential integral, and γ is the Euler constant. By the aid of the asymptotic expansion formula

$$E_i(z) \underset{z\to+\infty}{\sim} e^z/z,$$

one obtains

$$\mathcal{T}_{1,1}^{(3)}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))\, e^{\kappa^2(1-c)\Delta(x)}. \tag{3.10}$$

It is a striking fact that the formulae (3.2), (3.3), (3.8) and (3.10) can be taken together in a single one

$$\mathcal{T}_{1,1}^{(\nu)}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{(\nu-1)/2} e^{\kappa^2(1-c)\Delta(x)}, \qquad (\nu = 1, 2, 3, 4), \tag{3.11}$$

and this leads us to believe that (3.11) also holds for any positive natural number ν. Consequently, when one is only taking interest in knowing whether the arbitrary choice of gauge can be exploited to make the gravity superpropagator ambiguity-free, a simple calculation of the corresponding superpropagator in the scalar case $\nu = 1$, provides the answer. If on the other hand, one likes to obtain the complete leading asymptotic term, it seems sufficient to calculate this term for the corresponding superpropagator in the much simpler case $\nu = 2$, and to change afterwards the exponent of the power of $\Delta(x)$ in accordance with the formula (3.11). We have also evaluated the asymptotic behaviour of the functions $\mathcal{T}_{1,l}^{(\nu)}[\Delta]$ for $\nu = 1, 2, 4$ and have found

$$\mathcal{T}_{1,l}^{(\nu)}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{(\nu-1)/2} e^{l\kappa^2(1-c)\Delta(x)}, \qquad (\nu = 1, 2, 4),$$

which is in agreement with the previous remarks. Moreover, it is seen that the value $\frac{3}{2}$ for the exponent of the power of $\Delta(x)$ is typical for four-dimensional real symmetric matrix fields $\phi_{\alpha\beta}(x)$ satisfying (2.1).

Turning to the evaluation of the asymptotic behaviour of $\mathscr{E}^{(4)}_{k,l}[\Delta]$ one first calculates

$$\mathscr{E}^{(1)}_{k,l}[\Delta] = \mathscr{T}^{(1)}_{k,l}[\Delta] = \langle e^{k\kappa\phi(x)}, e^{l\kappa\phi(0)}\rangle = e^{kl\kappa^2(1-c)\Delta(x)}.$$

It is now an easy step to suggest that

$$\mathscr{T}^{(4)}_{k,l}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{3/2}\, e^{kl\kappa^2(1-c)\Delta(x)}, \tag{3.12}$$

or even more generally

$$\mathscr{T}^{(\nu)}_{k,l}[\Delta] \underset{\Delta\to+\infty}{\sim} (\kappa^2\Delta(x))^{(\nu-1)/2}\, e^{kl\kappa^2(1-c)\Delta(x)}. \tag{3.13}$$

4. Remarks

The results of the preceding section can be generalised to the case of N-point amplitudes. As an example we consider

$$\mathscr{T}^{(\nu)}_{1,1,1}[\Delta_{12}, \Delta_{23}, \Delta_{31}] \equiv \langle \mathrm{Tr}\, e^{\kappa\phi(x)}, \mathrm{Tr}\, e^{\kappa\phi(y)}, \mathrm{Tr}\, e^{\kappa\phi(z)}\rangle,$$

where ν is the dimension of the matrix field, and

$$\Delta_{12} \equiv \Delta(x-y), \qquad \Delta_{23} \equiv \Delta(y-z), \qquad \Delta_{31} \equiv \Delta(z-x).$$

For $\nu = 1$, one obtains

$$\mathscr{T}^{(1)}_{1,1,1}[\Delta_{12}, \Delta_{23}, \Delta_{31}] = \sum_{L=0}^{\infty}\sum_{M=0}^{\infty}\sum_{N=0}^{\infty} \frac{\kappa^{L+M+N}}{L!M!N!}\langle \phi^L(x), \phi^M(y), \phi^N(z)\rangle.$$

Defining

$$\langle \phi(y), \phi(z)\rangle = (1-c)\Delta(y-z) = (1-c)\Delta_{23}, \qquad \text{(cyclic permutation)} \tag{4.2}$$

$$\alpha = \frac{M+N-L}{2}, \qquad \beta = \frac{N+L-M}{2}, \qquad \gamma = \frac{L+M-N}{2}, \tag{4.3}$$

or

$$\alpha+\beta = N, \qquad \alpha+\gamma = M, \qquad \beta+\gamma = L, \tag{4.4}$$

it is easily verified that

$$\langle \phi^L(x), \phi^M(y), \phi^N(z)\rangle$$

$$= \begin{cases} \dfrac{L!M!N!}{\alpha!\beta!\gamma!}(1-c)^{\alpha+\beta+\gamma}\Delta_{23}^{\alpha}\Delta_{31}^{\beta}\Delta_{12}^{\gamma} & \text{if } L+M+N \text{ is even, and} \\ & |M-N| \leq L \leq M+N, \text{ (cyclic permutation)} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (4.1) reduces to

$$\mathcal{T}^{(1)}_{1,1,1}[\Delta_{12},\Delta_{23},\Delta_{31}]=\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{[(1-c)\kappa^2]^{\alpha+\beta+\gamma}}{\alpha!\beta!\gamma!}\Delta_{23}^{\alpha}\Delta_{31}^{\beta}\Delta_{12}^{\gamma}=\mathrm{e}^{\kappa^2(1-c)(\Delta_{23}+\Delta_{31}+\Delta_{12})}, \tag{4.5}$$

and one may conjecture that

$$\mathcal{T}^{(\nu)}_{1,1,1}[\Delta_{ij}]\underset{\Delta_{ij}\to+\infty}{\sim}(\kappa^2\Delta_{ij})^{(\nu-1)/2}\,\mathrm{e}^{\kappa^2(1-c)\Delta_{ij}}\qquad(i,j=1,2,3,\ i\neq j). \tag{4.6}$$

The generalisation to N-point amplitudes ($N>3$) is now trivial. It has to be noted that the validity of (3.13) and (4.6) for $\nu=2$ has been checked by direct calculation. A diagonalisation technique has been used to evaluate $\mathcal{T}^{(2)}_{k,l}[\Delta]$ and $\mathcal{T}^{(2)}_{1,1,1}[\Delta_{ij}]$. The reader is referred to a future publication for details.

Finally we want to comment on a striking relationship. The chiral superpropagator calculations of Delbourgo (1972) and Ashmore and Delbourgo (1973b) demonstrate that if the field $\phi_\alpha{}^\beta(x)$ is Hermitian and satisfies

$$\langle\phi_\alpha{}^\beta(x),\phi_\gamma{}^\delta(0)\rangle=\delta_\gamma{}^\beta\delta_\alpha{}^\delta\,\Delta(x), \tag{4.7}$$

the asymptotic behaviour of the superpropagators $\mathscr{C}^{(\nu)}_{1,1}[\Delta]$, which are the chiral analogues of $\mathscr{G}^{(\nu)}_{1,1}[\Delta]$, is given for SU(2) by

$$\mathscr{C}^{(2)}_{1,1}[\Delta]\underset{\Delta\to+\infty}{\sim}(\kappa^2\Delta(x))\,\mathrm{e}^{\kappa^2\Delta(x)}, \tag{4.8}$$

and for SU(3) by

$$\mathscr{C}^{(3)}_{1,1}[\Delta]\underset{\Delta\to+\infty}{\sim}(\kappa^2\Delta(x))^2\,\mathrm{e}^{\kappa^2\Delta(x)}, \tag{4.9}$$

if not all field components are real. In the latter case the asymptotic behaviour is exactly that of the gravity analogue. It is almost trivial that

$$\mathscr{C}^{(1)}_{1,1}[\Delta]\underset{\Delta\to+\infty}{\sim}\mathrm{e}^{\kappa^2\Delta(x)}, \tag{4.10}$$

and one is thus led to generalise (4.8)–(4.10) by the formula

$$\mathscr{C}^{(\nu)}_{1,1}[\Delta]\underset{\Delta\to+\infty}{\sim}(\kappa^2\Delta(x))^{\nu-1}\,\mathrm{e}^{\kappa^2\Delta(x)}. \tag{4.11}$$

Comparing this to (3.11) one notices that by imposing on the Hermitian fields the supplementary condition of reality, the exponent of the power of $\Delta(x)$ in the leading asymptotic term is divided by two. This relationship can be found at a different level in the Ashmore algorithm. Where for Hermitian fields there occurs a determinant of an antisymmetrical matrix, one finds at the same stage for real symmetrical fields a Pfaffian, which is essentially the square root of a determinant. These facts indicate that there might exist group-theoretical arguments to obtain (3.11) and (4.11) in a rigorous way.

Another important difference between chiral and gravity superpropagators is that for the latter the gauge parameter c (or the weight ω) can be given values that make the superpropagators free of ambiguities. As in our belief this manifest uniqueness in certain gauges should not contradict the usually supposed gauge and weight independent of physical amplitudes, it provides us with an indication that in other gauges too ambiguities might be avoided. Knowing that this is only achieved at the expense of a supplementary condition such as the Lehman–Pohlmeyer minimality *ansatz*, we can conclude that the invocation of such a condition in certain gauges becomes less artificial.

Acknowledgments

We like to express our gratitude to Professor C Grosjean for useful comments and his continuing interest during the course of this work. We also thank Dr R Delbourgo for some personal notes on the existence of a limiting procedure.

References

Ashmore J and Delbourgo R 1973a *J. Math. Phys.* **14** 176–81
—— 1973b *J. Math. Phys.* **14** 569–71
Delbourgo R 1972 *J. Math. Phys.* **13** 464–8
De Meyer H 1976 *J. Phys. A: Math. Gen.* **9** 1333–47
Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)
Isham C J, Salam A and Strathdee J 1971 *Phys. Rev.* D **3** 1805–17
—— 1972 *Phys. Rev.* D **5** 2548–65
—— 1973 *Phys. Lett.* **46B** 407–11
Kapoor A K 1976 *J. Math. Phys.* **17** 61–2
Von Siegel C L 1935 *Ann. Math., NY* **36** 527–606